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MODES OF FINITE RESPONSE TIME CONTROL*

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ABSTRACT

A linear autonomous system with a single control variable is considered. There are, in general, several modes of finite response time control for such a system. The concepts of single component regulation and multiple component regulation are defined. It is then shown that a multiple component regulation problem can be transformed into a single component regulation problem. Thus it is possible to express any of the modes of control considered as control of a single input, single output system.

AUTHOR

INTRODUCTION

The system considered is represented by the vector differential equation

$$\dot{x}(t) = Ax(t) + bu(t) \tag{1}$$

where dot denotes differentiation with respect to time, t,

- $x(t)$ is a column vector with elements $x_1(t), x_2(t), \ldots, x_n(t)$ which describe the state of the system,
- $u(t)$ is a scalar control variable,
- A is a constant nxn matrix, and
- b is a constant column vector.

It is assumed that the system (1) is completely controllable. This means that for any initial state of the system there exists a control defined on a closed finite interval of time, [0,T] such

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that the state of the system arrives at the zero state ($x=0$) at the time T. It is known (reference 3, pp. 483-484) that a necessary and sufficient condition for complete controllability of the system (1) is that the vectors $b, Ab, \ldots, A^{n-1}b$ are linearly independent, i.e.,

$$\det|b, Ab, \ldots, A^{n-1}| \neq 0 \tag{2}$$

Single component regulation is defined as control of the system such that one component of the state vector is transferred to zero in a finite time and held zero thereafter. Multiple component regulation is defined as control of the system such that more than one component of the state vector are transferred to zero in a finite time and held zero thereafter. As an example of a particular type of multiple component control a time optimal multiple component regulation problem could be defined when $u(t)$ is constrained in amplitude as follows: for any initial condition find a control satisfying the amplitude constraint on the interval $(0, \infty)$ such that the components to be controlled are transferred to zero in the minimum time such that they may be held at zero thereafter. The time optimal single component regulation problem was first discussed by Schmidt (reference 5, pp. 40-69) and was later treated by Harvey and Lee (references 1, 2, 4).

The definitions of single component and multiple component regulation given above are somewhat ambiguous and are not mutually exclusive. It is possible in some cases to state the same control problem as a single component or as a multiple component regulation problem. For example, consider the system

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + \begin{vmatrix} 0 \\ 1 \end{vmatrix} u$$

The single component regulation problem of controlling x_1 is the same as the multiple component regulation problem of controlling x_1 and x_2 since $x_2 = \dot{x}_1$ and a necessary condition for holding x_1 at zero is that x_2 be held at zero. Thus, whether this particular control problem is viewed as a single or multiple component regulation problem depends on the desire of the analyst.

The following section is devoted to a constructive proof of this paper's principal result:

Given a multiple component regulation problem, there exists a linear transformation of the state space such that the given problem is a single component regulation problem in the transformed state variables.

This result makes possible the application of the theory related to time-optimal single component regulation (references 1, 2, 4, 5) to time optimal multiple component regulation. Also, the result allows the control engineer faced with a multiple component regulation problem to reformulate the problem as a single input, single output problem with which he may have more familiarity.

DEVELOPMENT OF TRANSFORMATIONS

Consider the following multiple component regulation problem for the system (1). Suppose that the components x_1, x_2, ...,x_m, $1 \leq m \leq n$ are to be controlled, i.e., given an arbitrary initial condition $x(0) = x^o$, find a control $u(t)$, $0 \leq t$, depending on x^o,

such that the corresponding solution of (1) satisfies $x_1(t) = x_2(t) = \cdots = x_m(t) = 0$ for $t \geq \tau$ for some real number τ which may depend on x^o.

For convenience the following notation is introduced. The vector x will be partitioned into two vectors ξ_1 and ξ_2 with $\xi_1 = (x_1, x_2, \cdots, x_m)'$ and $\xi_2 = (x_{m+1}, x_{m+2}, \cdots, x_n)'$ where $'$ denotes transpose. Also the vector b will be partitioned into two vectors $\beta_1 = (b_1, b_2, \cdots, b_m)'$ and $\beta_2 = (b_{m+1}, b_{m+2}, \cdots, b_n)'$. The matrix A will be partitioned into four submatrices, A_1, A_2, A_3, and A_4 with $A_1 = |a_{ij}|$, $1 \leq i \leq m$, $1 \leq j \leq m$; $A_2 = |a_{ij}|$, $1 \leq i \leq m$, $m+1 \leq j \leq n$; $A_3 = |a_{ij}|$, $m+1 \leq i \leq n$, $1 \leq j \leq m$; $A_4 = |a_{ij}|$, $m+1 \leq i \leq n$, $m+1 \leq j \leq n$. Then the equation (1) can be written as

$$\begin{aligned} \dot{\xi}_1 &= A_1\xi_1 + A_2\xi_2 + \beta_1 u \\ \dot{\xi}_2 &= A_3\xi_1 + A_4\xi_2 + \beta_2 u \end{aligned} \tag{2}$$

The following theorem, which is evident from an examination of equation (2), is readily established:

Theorem 1. If the system (1) is completely controllable, then A_2 and β_1 are not both zero.

Proof: Suppose that A_2 and β_1 are both zero. Then it is easy to show that the vector $A^k b$ has zeros for its first m elements, with k a nonnegative integer. Thus the matrix $|b, Ab, \ldots, A^{n-1}b|$ has m rows of zeros and hence its determinant is zero.

It may occur, as in the example cited in the introduction, that the control of ξ_1 may imply the control of certain linear combinations of components of ξ_2. To examine this possibility, consider the requirement that $\xi_1(t) = 0$ for all $t \geq T$ for some time T. From the

system (2) it is clear that for $t \geq T$:

$$\begin{aligned} 0 &= A_2\xi_2 + \beta_1 u \\ \dot{\xi}_2 &= A_4\xi_2 + \beta_2 u \end{aligned} \tag{3}$$

If $\beta_1 = 0$ then $A_2\xi_2 = 0$ for $t \geq T$. Hence control to the subspace defined by $\xi_1 = 0$ implies control to the subspace, $\hat{\xi}_1 = 0$, defined by $\xi_1 = 0$ and $A_2\xi_2 = 0$. $\hat{\xi}_1$ may be obtained by adjoining to ξ_1 the linearly independent elements of $A_2\xi_2$. The problem may then be restated with $\hat{\xi}_1$ and $\hat{\xi}_2$ (the projection of x onto $\hat{\xi}_1 = 0$) replacing ξ_1 and ξ_2. The matrices A_1, A_2, A_3, A_4 and the vectors β_1 and β_2 would of course have to be replaced with corresponding matrices and vectors. In case $\beta_1 \neq 0$ it is clear from (3) that $u = -\beta_1' A_2\xi_2 / \|\beta_1\|^2$ and hence $(\|\beta_1\|^2 A_2 - \beta_1\beta_1' A_2)\xi_2 = 0$. As in the case when $\beta_1 = 0$ the problem can be reformulated with x partitioned into vectors $\hat{\xi}_1$ and $\hat{\xi}_2$. These procedures may be repeated until it is found that control to the subspace $\xi_1 = 0$ does not imply control to any smaller subspace. The number of reformulations is finite and is in fact less than or equal to n-m.

Now let us assume that theproblem stated at the beginning of this section is the result of necessary reformulations so that control to the subspace, $\xi_1 = 0$, does not imply control to any smaller subspace. This hypothesis guarantees that

$$\beta_1 \neq 0 \text{ and } A_2 = \beta_1\beta_1' A_2 / \|\beta_1\|^2. \tag{4}$$

To show this suppose that $\beta_1 = 0$. Then, since the system is assumed to be completely controllable, $A_2 \neq 0$ and control to the subspace, $\xi_1 = 0$, implies control to the smaller subspace, $\xi_1 = 0$ and $A_2\xi_2 = 0$,

which contradicts our hypothesis. Thus $\beta_1 \neq 0$ and hence $A_2 = \beta_1\beta_1' A_2 / \|\beta_1\|^2$, because if this were not the case control to the subspace, $\xi_1 = 0$, would imply control to the smaller subspace, $\xi_1 = 0$ and $(A_2 - \beta_1\beta_1' A_2 / \|\beta_1\|^2)\xi_2 = 0$ which contradicts our hypothesis.

With condition (4) established, the system (2) will be transformed into a particular form, in which it is evident that the problem is a single component control problem. Let $z = Sx$ where S is an nxn matrix partitioned into the submatrices S_1, S_2, S_3 and S_4 in the same manner that was used in partitioning A. The matrices S_2 and S_3 are zero matrices of appropriate size and S_4 is the $(n-m)^{th}$ order identity matrix. The matrix S_1 is defined indirectly by defining a matrix denoted by S_1^{-1} and the nonsingularity of S_1^{-1} is established in:

Theorem 2. If the system (1) is completely controllable and (4) is satisfied, then S_1^{-1} is nonsingular, where S_1^{-1} is defined as

$$S_1^{-1} = |A_1^{m-1}\beta_1, A_1^{m-2}\beta_1, \ldots, A_1\beta_1, \beta_1| .$$

The proof of this theorem will be given following the proof of theorem 3. Partitioning the vector z into m and n-m vectors ζ_1 and ζ_2, the transformation may be written as $\zeta_1 = S_1\xi_1$, $\zeta_2 = \xi_2$. The transformed system is

$$\begin{aligned} \dot{\zeta}_1 &= S_1 A_1 S_1^{-1}\zeta_1 + S_1 A_2 \zeta_2 + S_1\beta_1 u \\ \dot{\zeta}_2 &= A_3 S_1^{-1}\zeta_1 + A_4\zeta_2 + \beta_2 u \end{aligned} \tag{5}$$

The matrix S_1 has the property that $S_1\beta_1$ is a unit vector with its first m-1 elements zero. From this result and condition (4) it is clear that the first m-1 rows of S_1A_2 are zero and the last row is

$\beta_1' A_2 / \|\beta_1\|^2$. The matrix $S_1 A_1 S_1^{-1}$ has ones on the super diagonal, the first column is a vector c and all other elements are zero where the elements c_i satisfy

$$A_1{}^m = \sum_{i=1}^{m} c_i A_1^{m-i} .$$

From the form of (5) it is easy to establish:

Theorem 3. Regulation of z_1 (the first component of z) is equivalent to the regulation of ζ_1.

Proof: Clearly, regulation of ζ_1 implies regulation of z_1. From (5), $z_{k+1} = \dot{z}_k - c_k z_1$, $k = 1,2,\ldots,m-1$. Therefore

$$z_{k+1} = z_1^{(k)} - \sum_{j=o}^{k-1} c_{k-j}\, z_1^{(j)}$$

where $z_1^{(j)}$ denotes the j^{th} time derivative of z_1. Thus ζ_1 can be expressed in terms of z_1 and its first m-1 derivatives and hence regulation of z_1 implies regulation of ζ_1.

Proof of theorem 2. From the condition (4) it is clear that $A_2\beta$ is a multiple of β_1 for any n-m vector β. Let γ_{1j} and γ_{2j} denote m and n-m vectors respectively such that

$$A^j b = \begin{vmatrix} \gamma_{1j} \\ \gamma_{2j} \end{vmatrix} \quad \text{for each } j \geq 0.$$

By induction it can be shown that

$$\gamma_{1j} = \sum_{k=o}^{j} \lambda_k\, A_1^{j-k}\, \beta_1 \tag{6}$$

where λ_k is a scalar for $k=0,1,\ldots,j$, $\lambda_o = 1$ and $A_2\gamma_{2k} = \lambda_{k+1}\beta_1$. Denoting the matrix $|\beta_1, A_1\beta_1, \ldots, A_1^{m-1}\beta_1|$ by M and the matrix $|b, Ab, \ldots, A^{n-1}b|$ by N, the determinant of N may be written as:

$$\det \begin{vmatrix} \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1n} \\ \gamma_{20} & \gamma_{21} & \cdots & \gamma_{2n} \end{vmatrix}.$$ Using (6), the Cayley-Hamilton theorem and the elementary properties of determinants, this determinant may be written as

$$\det \begin{vmatrix} M & O \\ P & Q \end{vmatrix}$$ where O is the m x n - m matrix of zeros. Thus the determinant of N is the product of the determinants of M and Q. The determinant of N is non zero since the system (1) is assumed to be completely controllable and hence the determinant of M is nonzero. But the determinant of M is the determinant of S_1^{-1}, so that S_1^{-1} is nonsingular.

REMARKS

If ζ_1 is to be held zero after the response time T it is clear from (5) that for $t \geq T$:

$$u(t) = -\beta_1' A_2 \zeta_2(t) / \|\beta_1\|^2 \tag{7}$$

and

$$\dot{\zeta}_2 = (A_4 - \beta_2\beta_1' A_2 / \|\beta_1\|^2)\, \zeta_2 \tag{8}$$

If the control u(t) is required to satisfy the constraint $|u(t)| \leq 1$ for all t, it is necessary to consider u(t) given by (7) and (8) with $\zeta_2(T)$ being the initial condition for (8). Satisfying the constraint imposes constraints on the initial condition $\zeta_2(T)$. It may occur that some constraints **are** of the form $\eta'\zeta_2(T) = 0$ where η is a constant n-m vector. In this case the control of ζ_1 implies the control to the subspace, $\zeta_1 = 0$, $\eta'\zeta_2 = 0$, and the problem may then be reformulated to be control to this subspace.

CONCLUSIONS

It has been shown that multiple component regulation problems can be transformed into single component regulation problems for linear constant coefficient systems with a scalar control input. This permits one to view such problems as single input, single output control problems. The development presented is of a constructive nature so that the single output of the single component formulation of the regulation problem may be determined explicitly.

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